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1991 J. Phys. A: Math. Gen. 24 3817

(http://iopscience.iop.org/0305-4470/24/16/019)

J. Phys. A: Math. Gen. 24 (1991) 3817-3823. Printed in the UK

Flux quantization and quantum mechanics on Riemann surfaces in an external magnetic field

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Received 9 October 1990

Abstract. We investigate the possibility of applying an external constant magnetic field to a quantum mechanical system consisting of a particle moving on a compact or non-compact two-dimensional manifold of constant negative Gaussian curvature and of finite volume. For the motion on compact Riemann surfaces we find that a consistent formulation is only possible if the magnetic flux is quantized, as it is proportional to the (integrated) first Chern class of a certain complex line bundle over the manifold. In the case of non-compact surfaces of finite volume we obtain the striking result that the magnetic flux has to vanish identically due to the theorem that any holomorphic line bundle over a non-compact Riemann surface is holomorphically trivial.

1. Introduction

The free motion of a particle on two-dimensional surfaces has recently attracted some attention [1-3], as the classical dynamics of such systems is chaotic whenever the surface is endowed with a metric of negative curvature. The case of a compact surface of genus two turns out to be one of the simplest examples of a chaotic Hamiltonian system. Its quantum mechanical version (the Hadamard-Gutzwiller model) leads to the eigenvalue problem for the corresponding Laplace-Beltrami operator. This is a nice example to study the quantum mechanical properties of a classically chaotic system. Non-compact surfaces, that may have cusps and thus extend to infinity, can be used to investigate chaotic scattering processes [2].

It is tempting to ask what happens when the motion of the particle is no longer free but takes place in an external electromagnetic field. The simplest case will be a constant magnetic field. If the surface is a plane, there are bound states formed, the so-called Landau levels, under the influence of a magnetic field. A similar phenomenon also takes place on the surfaces considered here. Our main purpose in this paper is to find out what restrictions on the value of the magnetic field are imposed and thus what models can be consistently defined. We will see that in certain cases the magnetic flux is quantized, whereas in other cases there is no constant magnetic field allowed

§ Supported by Doktorandenstipendium der Universität Hamburg.

‖ On sabbatical leave from II. Institut für Theoretische Physik, Universität Hamburg.

¶ Unité de Recherche de l'Ecole Normale Supérieure et de l'Université Pierre et Marie Curie, associée au CNRS.

at all. With our discussion we want to clarify some problems that occurred in recent studies of scattering systems under the influence of magnetic fields.

The organization of this paper is as follows. First we review the treatment of electromagnetic fields on arbitrary manifolds and define what we call a constant magnetic field. Then we present the models we want to discuss, namely those that are interesting for a study of the quantum mechanical behaviour of a classically chaotic system and allow an application of Selberg's trace formula, which serves as an exact periodic-orbit formula. For these models we determine the allowed values for the constant magnetic field. In the final section we comment on a recent proposal to define chaotic scattering systems in constant magnetic fields.

2. Electrodynamics on general manifolds

If one wants to discuss electrodynamics on a topologically non-trivial smooth configuration manifold M, one has to treat the global properties of the fields and state vectors very carefully. The field strength F and the vector potential A have to be considered as curvature and connection forms of a principal U(1)-bundle on M. In general it is not possible to define a vector potential A as a global one-form, a fact that is well known, e.g. from the treatment of Dirac's monopole or the Aharonov–Bohm effect.

In this paper we discuss a special case. M will be a two-dimensional surface endowed with a Riemannian metric of constant negative curvature and of finite volume. F will represent a constant magnetic field, a notion to be defined properly later.

In quantum mechanics one generally describes electromagnetism in terms of principal U(1)-bundles P over M (see e.g. [4]). The field strength F is a two-form on M that is closed due to one half of Maxwell's equations, $\mathrm{d}F=0$, and thus defines a DeRham cohomology class in $H^2(M,\mathbb{R})$. If, in addition, F is exact, there exists a globally defined vector potential A, $F=\mathrm{d}A$. Besides these zero-cohomological field strengths it is also possible to define an electromagnetic theory if only F represents an integral cohomology class, $[F]\in H^2(M,\mathbb{Z})$. This fact is due to the following theorem (see e.g. [5]).

Theorem. Let ω be a two-form on M. Iff $[(1/2\pi\mathrm{i})\omega]\in H^2(M,\mathbb{Z})$, then there exists a complex line bundle L over M with connection ∇, such that ω is the curvature form associated with ∇.

Under these conditions the state vectors ψ of the system are sections in L and $F=(1/\mathrm{i}e)\omega$ is the field strength. The connection ∇ in local coordinates $\{x^\mu_{(k)}\}$ looks like $\nabla^{(k)}_\mu=\partial^{(k)}_\mu+\mathrm{i}eA^{(k)}_\mu$, where $A^{(k)}$ is the *local* vector potential in a given gauge. The stationary Schrödinger equation to be solved is then

$$-\nabla^2\psi=E\psi. \tag{1}$$

The magnetic flux Φ through M reads

$$\Phi=e\int_M F=-\mathrm{i}\int_M \omega=2\pi c(L)\in 2\pi\mathbb{Z} \tag{2}$$

where $c(L)$ is the (integrated) first Chern class of the line bundle L, which is an integer number.

To define what we want to call a constant magnetic field on a two-dimensional manifold we first review the case of a constant magnetic field on a surface S that is embedded in $\mathbb{R}^3$. There the magnetic field is just a vector $\boldsymbol{B}$ in $\mathbb{R}^3$. S carries a metric induced from the Euclidean metric in $\mathbb{R}^3$, $g_{\mu\nu} := \partial \mathbf{x}/\partial u^\mu \cdot \partial \mathbf{x}/\partial u^\nu$, where S is given by the parametrization $\boldsymbol{x}(u^1, u^2)$. The embedding in $\mathbb{R}^3$ now defines an orientation of S, given by the outward normal vector field $\boldsymbol{N} := \partial_1 \boldsymbol{x} \wedge \partial_2 \boldsymbol{x} / |\partial_1 \boldsymbol{x} \wedge \partial_2 \boldsymbol{x}|$. Therefore one can define an 'oriented volume form' $\mathrm{d}\boldsymbol{f} := \boldsymbol{N}\sqrt{g}\, \mathrm{d}u^1 \wedge \mathrm{d}u^2 = \boldsymbol{N}\, \mathrm{d(vol)}$.

The magnetic flux of $\mathbf{B}$ through S thus is

$$\Phi = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{f} = |\boldsymbol{B}| \int_S \cos\theta \, \mathrm{d(vol)} \tag{3}$$

where θ is the angle between $\boldsymbol{B}$ and $\boldsymbol{N}$.

When there is no embedding of the manifold M in some higher-dimensional manifold (like $\mathbb{R}^3$) there is no orientation of M. Thus the most natural way to define a constant magnetic field seems to be the following one (in local coordinates). When a Riemannian metric $\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ is given, then there exists the volume form $\mathrm{d(vol)} = \sqrt{g}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$, $g := \det(g_{\mu\nu})$. Then put $F := B\, \mathrm{d(vol)}$, $B \in \mathbb{R}$ is the magnitude of the magnetic field. For a manifold of finite volume V the flux through M is

$$\Phi = e \int_M F = eB \int_M \mathrm{d(vol)} = eBV. \tag{4}$$

As $\Phi = 2\pi c(L)$, see equation (2), one obtains a quantization of the flux and the value of the magnetic field, respectively,

$$B = \frac{2\pi c(L)}{eV} \qquad c(L) \in \mathbb{Z}. \tag{5}$$

3. The models

We consider Riemann surfaces that can be uniformized in the Poincaré upper half-plane $\mathcal{H} = \{z = x + \mathrm{i}y \mid y > 0\}$ with metric $\mathrm{d}s^2 = y^{-2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$ of constant negative Gaussian curvature $K = -1$. Among these are all compact Riemann surfaces of genus $g \geqslant 2$ and some non-compact surfaces, as e.g. the fundamental domain of the modular group $\mathrm{PSL}(2, \mathbb{Z})$, and the respective fundamental domains of the subgroups of the modular group. One of these is the so-called leaky torus [2], which is a surface of genus one with one cusp. The study of the quantum mechanical motion of a particle on such surfaces has been initiated by Gutzwiller [1, 2] and was continued by others [3, 6].

In general all these surfaces can be represented as a fundamental domain in $\mathcal{H}$ of a discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ (called a Fuchsian group). Any element $\gamma \in \Gamma$ can be represented by a 2×2 matrix

$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc = 1, \tag{6}$$

where $\mathcal{M}$ and $-\mathcal{M}$ have to be identified as $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R}) / \{\pm \mathbb{1}\}$. Γ operates on $\mathcal{H}$ via fractional linear transformations,

$$\gamma z = \frac{az + b}{cz + d}. \tag{7}$$

Then $\Gamma\backslash\mathscr{H}$ is a Riemann surface. Any compact Riemann surface of genus $g \geqslant 2$ can be realized in this way due to the uniformization theorem, with Γ consisting only of hyperbolic transformations, that is $|a+d|>2$. Other examples of Fuchsian groups are, as mentioned above, the modular group and its subgroups, especially the congruence subgroups (see e.g. [7]). They yield Riemann surfaces $\Gamma\backslash\mathscr{H}$ with cusps and of various genera. Thus we have a large variety of different interesting models at hand.

In the case of free particles the state vectors are (smooth) functions on $M=\Gamma\backslash\mathscr{H}$. Equivalently one can take Γ-automorphic functions on $\mathscr{H}$, that is $\psi:\mathscr{H}\to\mathbb{C}$, $\psi(\gamma z)=\psi(z)$ for all $\gamma\in\Gamma$. When a constant magnetic field is present the state vectors are sections in an appropriate bundle, as discussed in the previous section. They can also be represented as functions on $\mathscr{H}$, although not Γ-automorphic ones of course. The behaviour of such a function $\psi:\mathscr{H}\to\mathbb{C}$ under a transformation $z\mapsto\gamma z$, $\gamma\in\Gamma$, has been derived in [8]. There the authors choose the gauge $A=A_x\,\mathrm{d}x+A_y\,\mathrm{d}y$, $A_x=-B/y$, $A_y=0$. Then they show that the state vector, viewed as a function on $\mathscr{H}$, transforms like

$$\psi(\gamma z)=\psi(z)\exp\{\mathrm{i}2eB\arg(cz+d)\}=\left(\frac{cz+d}{|cz+d|}\right)^{2eB}\psi(z). \tag{8}$$

From this transformation rule it seems to be obvious that ψ defines a section in some line bundle over M and not a function on M, but this has to be checked of course, since the bundle may be trival. This bundle can be related to the canonical bundle $K=T^*_{(1,0)}M$ of holomorphic one-forms on M. A holomorphic function $f:\mathscr{H}\to\mathbb{C}$, that obeys the transformation rule (8), can be associated with a section in K^λ, $\lambda=eB$, in the following way. Form $g(z):=y^{-\lambda}f(z)$, then

$$g(\gamma z)=(cz+d)^{2\lambda}g(z) \qquad \forall\gamma\in\Gamma. \tag{9}$$

This defines a global section in K^λ, since $\mathrm{d}(\gamma z)/\mathrm{d}z=(cz+d)^{-2}$, therefore $g(\gamma z)\,\mathrm{d}(\gamma z)^\lambda=g(z)\,\mathrm{d}z^\lambda$. f itself defines a section in $K^{\lambda/2}\otimes\bar{K}^{-\lambda/2}$. Functions of the same type as g, i.e. that satisfy (9), are called Γ-automorphic forms of weight 2λ. (For the connection between automorphic forms and line bundles in general see [9].)

As a result we get that the state vector of a particle in a constant magnetic field defines a section in the bundle K^λ, $\lambda=eB$. As only for $\lambda\in\frac{1}{2}\mathbb{Z}$ such powers (in the sense of tensor products) of line bundles exist, this gives a quantization condition for the value of the magnetic field,

$$B=\frac{1}{2e}n \qquad n\in\mathbb{Z}. \tag{10}$$

This condition is necessary, but not yet sufficient.

For a compact surface of genus $g\geqslant 2$ the Gauss–Bonnet theorem yields $V=4\pi(g-1)$ for the volume of M. Therefore $(1/2\pi\mathrm{i})\omega$, $\omega=\mathrm{i}eF=\mathrm{i}eB\,\mathrm{d(vol)}$, is of integral cohomology class,

$$\frac{1}{2\pi\mathrm{i}}\int_M\omega=\frac{eB}{2\pi}V=n(g-1)\in\mathbb{Z}. \tag{11}$$

The theorem quoted in section 2 then guarantees the existence of the appropriate line bundle. It is, of course, just $L=K^{n/2}$. Thus for any value of $B=n/2e$, $n\in\mathbb{Z}$, there exists the corresponding physical system, therefore (10) is also a sufficient condition in the case of compact surfaces, and since the expression (11) equals $c(L)$ (see (2)), it is obvious that the respective bundles are non-trivial as long as $B\neq 0$.

Now we consider the case of non-compact surfaces (but of finite volume). There is the following theorem (see e.g. [10]).

Theorem. Any holomorphic line bundle over a non-compact Riemann surface is holomorphically trivial.

A trivial line bundle L, however, has a vanishing first Chern-class, $c(L)=0$. Since (5) says that B is proportional to $c(L)$, B also has to vanish, as V is finite. The conclusion thus is that it is impossible to apply a constant magnetic field, in the sense of section 2, to a particle moving on a non-compact Riemann surface of finite volume which has a uniformization in the Poincaré upper half-plane.

The Hamiltonian of the magnetic systems considered here (see (1) and [11]) is

$$H_B = -y^2(\partial_x^2+\partial_y^2) - \mathrm{i}2eBy\partial_x + e^2B^2. \tag{12}$$

The operators $D_{2\lambda} = -y^2(\partial_x^2+\partial_y^2) + 2\mathrm{i}\lambda y\partial_x$, acting on functions on $\mathscr{H}$ satisfying (8), are well known in the mathematical literature (see e.g. [12]). There is a Selberg trace formula for these operators (see e.g. [13]). This relates the eigenvalues of $D_{2\lambda}$ to the length spectrum of closed geodesics on $\Gamma\backslash\mathscr{H}$. Let $\rho_k = \frac{1}{4}+p_k^2$ denote the eigenvalues of $D_{2\lambda}$, then for compact Riemann surfaces with genus greater than one

$$\sum_{n=1}^{\infty} h(p_n) = \frac{V}{4\pi}\int_{-\infty}^{+\infty} \mathrm{d}p\, ph(p)\frac{\sinh(2\pi p)}{\cosh(2\pi p)+\cos(2\pi\lambda)} + \frac{1}{2}\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{\chi(\gamma)^{2\lambda k}l(\gamma)}{\sinh(kl(\gamma)/2)}g(kl(\gamma))$$
$$+\frac{V}{4\pi}\sum_{0\leq m<|\lambda|-1/2}(2|\lambda|-2m-1)h\left(\frac{\mathrm{i}}{2}(2|\lambda|-2m-1)\right). \tag{13}$$

χ is a multiplier system for the double cover $\hat{\Gamma}$ of Γ, $\Gamma=\hat{\Gamma}/\{\pm\mathbb{1}\}$, and $h(p)$ is any even function that is holomorphic in the strip $|\mathrm{Im}\, p|\leq\frac{1}{2}+\varepsilon$, $\varepsilon>0$, and decreases faster than $|p|^{-2}$ at infinity. $g(x)=\int_{-\infty}^{+\infty}(\mathrm{d}p/2\pi)h(p)\,\mathrm{e}^{\mathrm{i}px}$ is the Fourier transform of $h(p)$; $\{\gamma\}_p$ denotes all primitive conjugacy classes in Γ and $l(\gamma)$ is the length of the closed geodesic that corresponds to $\{\gamma\}_p$. The last term on the RHS of (13) represents a sum over the additional eigenvalues of $D_{2\lambda}$ for $\lambda\neq 0$, which can be viewed as Landau levels [11], when one considers the Hamiltonian (12). These eigenvalues are [12, 13]

$$(|\lambda|-m)(1+m-|\lambda|) \qquad 0\leq m<|\lambda|-\tfrac{1}{2}. \tag{14}$$

The trace formula (13) enables one to compute the eigenvalues of $D_{2\lambda}$ and thus those of $H_B = D_{-eB}+e^2B^2$, once one knows the geodesic length spectrum $\{l(\gamma)\,|\,\gamma\in\Gamma\}$ of M. Such a procedure can be applied to compute the energy-levels of a general chaotic Hamiltonian system in a semiclassical approximation and is known as Gutzwiller's periodic-orbit quantization [14]. Whenever one has a system where one can apply a Selberg trace formula, this periodic-orbit quantization becomes exact, i.e. it is not only valid as a semiclassical approximation. In the case $B=0$ and for genus two (Hadamard-Gutzwiller model) this method has proved useful to compute low-lying energies [3]. There one takes a function $h(p)$ that exhibits sharp peaks at the p_k, e.g. a Gaussian smearing

$$h(p) = \mathrm{e}^{-(p-p')^2/\varepsilon^2} + \mathrm{e}^{-(p+p')^2/\varepsilon^2}.$$

Thus also these magnetic systems may be quantized using the trace formula.

4. Discussion

The surfaces discussed in section 3 that have cusps, i.e. where Γ contains parabolic transformations, extend to infinity ($i\infty$ is a fixed point of the parabolic transformations). Therefore the phase shift in the wavefunction of a particle coming from $i\infty$ and being reflected back to infinity can be related to the scattering matrix $S(k)$, where k is the particle's momentum, $E = k^2+\frac{1}{4}$. Gutzwiller [2] discussed the S-matrix for the leaky torus and draw attention to the fact that the irregular behaviour of the phase shift, which is governed by the Riemann zeta function, is a clear signature of quantum chaos. Recently [6] the poles of the S-matrix which are given by the non-trivial zeros of the Riemann zeta function have been interpreted as resonances. (These poles lie in the unphysical sheet of the complex energy-plane, since the momenta have a negative imaginary part). Numerically this interpretation has been shown to be nicely consistent [6].

It is interesting to investigate the potential influence of a constant magnetic field on these scattering systems. In [8] it was claimed that the S-matrix gets modified by a factor that exhibits poles at the momenta associated with the Landau levels, i.e. those discrete eigenvalues of the Hamiltonian that are due to the presence of the magnetic field.

However, our result derived in section 3 implies that it is impossible to apply a constant magnetic field to non-compact Riemann surfaces of finite volume. Thus for the modular domain and the leaky torus there is no consistent formulation with a constant magnetic field, even if the magnetic field is quantized, contrary to the statement made in ref. [8].

Nevertheless these systems may be studied from a mathematical point of view. Although the line bundles in which the automorphic forms define sections are trivial, there may very well exist non-trivial automorphic forms. It is only that the triviality cannot be seen directly by inspecting the automorphic forms. The scattering states that were proposed in [8] are thus mathematically well-defined objects, they may be called 'generalized Eisenstein series' and are well known in the mathematical literature (see e.g. [12, 13]). There exists also a Selberg trace formula for these systems [13]. But the physical interpretation of these systems as chaotic scattering models in an external magnetic field is not consistent.

Acknowledgments

It is a pleasure to thank Holger Ninnemann for valuable discussions, especially for drawing our attention to the theorem quoted in section 3. One of us (FS) would like to thank Professors B Cagnac and J Dupont-Roc for the kind hospitality at the Laboratoire de Spectroscopie Hertzienne.

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